Theory of Magic Series

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This theory enables an inductive counting of the numbers of magic series.

Definition 1 (Kraitchik, 1942)

A set of **n** distinct integers taken from the interval **[1, n^2]** form a **magic series of order n** if their sum is the $\mathbf{n}^{\text{th}}$ magic constant $M_n = \frac{1}{2}n^2(n+1)$.

(For example {2, 8, 9, 15} is a magic series of order 4 since 2 + 8 + 9 + 15 = 34 .)

Definition 2

Let k, u, s be positive integers.
Define **N(k, u, s)** as number of sets $\{i_1, i_2, i_3, \ldots, i_k\}$ of **k** distinct integers, that fulfill both of the following two conditions:

(1) $0 < i_1 < i_2 < i_3 < \ldots < i_k \le \mathbf{u}$

(2) $i_1 + i_2 + i_3 + \ldots + i_k = \mathbf{s}$.

Proposition 1

$$\boxed{\text{The number of magic series of order n equals } N(n, n^2, M_n)}$$

Proof: Follows directly from the definitions.

Proposition 2 (not essential)

$$\boxed{N(k, u, s) > 0 \iff \tfrac{1}{2} k(k+1) \le s \le \tfrac{1}{2} k(2u-k+1)}$$

Proof: Since condition (2) of definition 2 only can be fulfilled if s is not smaller than the sum of the first k natural integers and not greater than the sum of the last k integers of the interval [1, u] . You can find at least one proper set of integers if s meets the conditions on the righthand side.

Proposition 3 (not essential)

$$\boxed{s - \tfrac{1}{2} k(k-1) \le u_1, u_2 \ \Rightarrow\ N(k, u_1, s) = N(k, u_2, s)}$$

Proof: If u is large enough then the last integer i_k is maximal if $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, … , $i_{k-1} = k-1$. In this case the value of i_k equals $s - \frac{1}{2}k(k-1)$. If u is not less than this value then the number of sets is independent of u.

Theorem 1

Calculation of **N(k, u, s)** with k = 1 :

$$\boxed{\begin{array}{c} u < s \iff N(1, u, s) = 0 \\ u \ge s \iff N(1, u, s) = 1 \end{array}}$$

Proof: For k=1 there only exists one possible set $\{i_1\} = \{s\}$ that meets condition (2). Condition (1) is not fulfilled if u is less than s.

Theorem 2

Calculation of **N(k, u, s)** with k > 2 using values of the type **N(k–1**, x, y) :

$$\boxed{N(k, u, s) = \sum_{w=k}^{\min(u, s-1)} N(k-1, w-1, s-w)}$$

Proof: Take away the largest integer $w = i_k$ of a considered set then the remaining k–1 integers are from the interval [1, w–1] and sum up to s–w. We just have to add the values N(k–1, w–1, s–w) for all possible largest numbers w ≤ u. The summation starts with w = k since the largest value of a set cannot be smaller than k. The summation ends at w = u or at w = s – 1 to avoid that (s – w) becomes zero or negative. [The summation may even end at w = s – ½ k(k–1) if this value is smaller than u (Proposition 2).]

Algorithm

For fixed k the values of N(k, u, s) may be stored in one array say No(u, s). Fill the array with the values for k = 1 according to Theorem 1. Calculate the values of N(2, u, s) according to Theorem 2 starting with the largest s and store the results for all values of u in the same array. To avoid disturbations it is neccessary to start with the largest s and decrement s successively. Propositions 2 and 3 may be applied to decrease the amount of calculations. At the end of this procedure No(u, s) will contain the values of N(2, u, s). Continue with next k until you reach the desired order n. After each step the number of magic series of order k may be saved (Proposition 1).
Note that the size of the array depends on the highest considered order n, therefore declare No(1 ... n^2, 1 ... M_n). If you want to get exact results, you have to use integer variables with high accuracy, in the case of n = 32 about 192 Bit per integer and about 400 MB for the complete array.

Results

Two <u>independent</u> calculations were done using 192-bit-integer and 64-bit-floatingpoint variables.

Order	Exact number of magic series	Floatingpoint value
01	1	1,00000000000000 E+00
02	2	2,00000000000000 E+00
03	8	8,00000000000000 E+00
04	86	8,60000000000000 E+01
05	1394	1,39400000000000 E+03
06	32134	3,21340000000000 E+04
07	957332	9,57332000000000 E+05
08	35154340	3,51543400000000 E+07
09	1537408202	1,53740820200000 E+09
10	78132541528	7,81325415280000 E+10
11	4528684996756	4,52868499675600 E+12
12	295011186006282	2,95011186006282 E+14
13	21345627856836734	2,13456278568367 E+16
14	1698954263159544138	1,69895426315954 E+18
15	147553846727480002824	1,47553846727480 E+20
16	13888244935445960871352	1,38882449354460 E+22
17	1408407905312396429259944	1,40840790531240 E+24
18	153105374581396386625831530	1,53105374581396 E+26
19	17762616557326928950637660912	1,77626165573269 E+28
20	2190684864446863915195866500356	2,19068486444686 E+30
21	286221079001041327793634043938470	2,86221079001041 E+32
22	39493409270082248457567923104977298	3,94934092700822 E+34
23	5739019677324553608481368828138484550	5,73901967732455 E+36
24	876085202984795348523051418634128837562	8,76085202984795 E+38
25	140170526450793924490478768121814869629364	1,40170526450794 E+41
26	23456461153390020211328759135664689342531028	2,34564611533900 E+43
27	4097641100787806775815644958425464097739938654	4,09764110078781 E+45
28	745947846718066619823209422870621836022069177558	7,45947846718067 E+47
29	141280774936453250057100993123755087750662375504136	1,41280774936453 E+50
30	27797610141981037322555479186167243505129073097363174	2,77976101419810 E+52
31	5673858009208148397135070998960708533898456476297052346	5,67385800920815 E+54
32	1199872454897380013845796517790093662180055383301098878668	1,19987245489738 E+57